

Anticommutativity of Skew-symmetric Elements under Generalized Oriented Involutions

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Abstract

Let R be a ring with $\text{char}(R) \neq 2$ whose unit group are denoted by $\mathcal{U}(R)$, G a group, and RG its group ring. Let $*$ be an involution in G , $\sigma : G \rightarrow \mathcal{U}(R)$ be a nontrivial group homomorphism, with $\ker \sigma = N$, satisfying $xx^* \in N$ for all $x \in G$, and define the generalized oriented involution $\sigma*$ in RG by $(\sum_{x \in G} \alpha_x x)^{\sigma*} = \sum_{x \in G} \sigma(x) \alpha_x x^*$. An element $\alpha \in RG$ is called skew-symmetric if $\alpha^{\sigma*} = -\alpha$, and the set of all skew-symmetric elements are denoted by $(RG)^-$. In this paper, we will classify the group rings RG such that $(RG)^-$ is anticommutative, generalizing, and obtaining as consequence, the main result of [GP13a].

Keywords: Group rings, rings with involution, skew-symmetric elements, generalized oriented involution, polynomial identity.

1 Introduction

Let RG be a group ring of a group G over a commutative ring R with unity. Given $*$, an involution in G , we can naturally induce an involution in RG , defined by the linear extension of $*$.

In a ring R with involution $*$, we call skew-symmetric elements those $r \in R$ such that $r^* = -r$ and collect them in the set $(RG)^-$. In the same way, we collect the symmetric elements, $r \in R$ such that $r^* = r$, in the set $(RG)^+$. Many papers classify the group rings in which these sets, defined with the linear extension of a group involution, satisfy a polynomial identity, see [BJPM09, GP13a, JM06, LSS09], and when a polynomial identity satisfied in these sets could be lifted to the entire group ring, see [GPS09].

Using an homomorphism $\sigma : G \rightarrow \{\mathcal{U}(R)\}$, we can define $\sigma* : RG \rightarrow RG$ mapping $\sum_{x \in G} \alpha_x x \mapsto \sum_{x \in G} \alpha_x \sigma(x) x^*$; easily we can check this map is an involution if, and only if, $x^* x^{-1} \in \ker \sigma$. When σ is nontrivial, $\sigma*$ is called a generalized oriented involution.

In the particular case $\sigma(G) = \{\pm 1\}$, $\sigma*$ is simply called an oriented involution and the papers [BP06, BJPM09, GP13b, GP14, CP12], searching for a generalization, study the identities stated above under this involution. To achieve these results, the authors used the information of the subgroup $N \leq G$, that was given by the similar results under the linearly extended involution.

Similarly to the fact that results on linear extension give information about $N \leq G$, the special case $\sigma(G) = \{\pm 1\}$ does the same regarding the subgroup $\sigma^{-1}(\{\pm 1\})$, that is C . For instance, the description of C given in [GP14] was relevant to the authors in order to obtain the main theorem in [PT15], and, although such a description in [CP12] was not actually used in [V13], the techniques developed in the first article were often used in the second one.

We say that a group G has an unique nontrivial commutator s , if $G' = \{1, s\}$. This class of group is quite frequent in the study of commutativity and anticommutativity of symmetric and skew-symmetric elements; some examples can be found in [BP06, BJPM09, GP13a, GP14, JM06, PT15],

where the involution, in this case, is given by $x^* = \{x, sx\}$, $\forall x \in G$. A group G is said to have limited commutativity, LC-group for short, if, given $x, y \in G$ such that $(x, y) = 1$, then x, y or $xy \in \mathcal{Z}(G)$, where $\mathcal{Z}(G)$ is the center of G . A special involution is naturally endowed in a LC-group G that has an unique nontrivial commutator s , namely, $*$: $G \rightarrow G$ mapping

$$x^* = \begin{cases} x, & \text{if } x \in \mathcal{Z}(G) \\ sx, & \text{contrary case;} \end{cases}$$

obviously this is a particular case of the previous one. We can verify this map defines an involution only in this group class and, in this case, we say that G is a SLC-group with canonical involution $*$. A better description of SLC-groups is given by [JM06, Theorem 2.4].

The groups with involution mentioned above shall play an important role in the our main theorem. The goal of this result is to classify the group rings RG in which the set of skew-symmetric elements, over a generalized oriented involution, is anticommutative. In order to do that, we deal with a similar argument used in [PT15], however we simply use the N description, instead of that of C , to obtain the result; this way we also reach the description of C as a consequence.

The kernel of σ will be denoted by N and the center of G , by $\mathcal{Z}(G)$. The symmetric elements under $*$ in G are collected in G_* ; $(x, y) = x^{-1}y^{-1}xy$ is the multiplicative commutator; and $x^y = y^{-1}xy$ is the conjugation of x by y . We will denote $N_* = G_* \cap N$ and $A' = \langle (x, y) : x, y \in A \rangle$. Throughout this work, we will assume that $\text{char}(R) \neq 2$ and will use this fact without further mention.

2 Skew-symmetric Elements Anticommute

In order to prove that $(RG)^-$ is anticommutative, it is enough to do that to a set of generators of $(RG)^-$.

For $\alpha \in RG$, write

$$\alpha = \sum_{x \in N_*} \alpha_x x + \sum_{x \in N \setminus G_*} \alpha_x x + \sum_{x \in G_* \setminus N} \alpha_x x + \sum_{x \in G \setminus (G_* \cup N)} \alpha_x x,$$

then

$$\alpha^{\sigma*} = \sum_{x \in N_*} \alpha_x x + \sum_{x \in N \setminus G_*} \alpha_x x^* + \sum_{x \in G_* \setminus N} \sigma(x) \alpha_x x + \sum_{x \in G \setminus (G_* \cup N)} \sigma(x) \alpha_x x^*,$$

so

$$\alpha^{\sigma*} = -\alpha \text{ if and only if } \begin{cases} \alpha_x = -\alpha_{x^*} & \text{for } x \in N_* \\ \alpha_x = -\alpha_{x^*} & \text{for } x \in N \setminus G_* \\ \alpha_x = -\sigma(x) \alpha_x & \text{for } x \in G_* \setminus N \\ \alpha_x = -\sigma(x) \alpha_{x^*} & \text{for } x \in G \setminus (G_* \cup N). \end{cases}$$

Thus $(RG)^-$ is spanned over R by elements in the sets,

$$\mathcal{A}_1 = \{\alpha x : x \in G_* \text{ e } \alpha(1 + \sigma(x)) = 0\}$$

$$\mathcal{A}_2 = \{x - \sigma(x)x^* : x \in G \setminus G_*\}.$$

Since N is invariant under $*$, we can easily verify that $\sigma *|_{RN} = *|_{RN}$, therefore, the oriented involution acts in RN as an ordinary involution, so we can deduce some information about N using the following result.

Theorem 2.1 (Theorem 2.2, [GP13a]). *Let G be a group with involution $*$ and let R be a ring of $\text{char}(R) \neq 2$. Then the set $(RG)^-$ is anticommutative if, and only if, either:*

- (1) G is abelian and $*$ = Id .
- (2) $\text{char}(R) = 4$, G is abelian, and exists $s \in G$ with $s^2 = 1$ and $x^* \in \{x, sx\}$, $\forall x \in G$.
- (3) $\text{char}(R) = 4$, G is a nonabelian group with a unique nonidentity commutator s , and $x^* \in \{x, sx\}$, $\forall x \in G$.

Lemma 2.2. Suppose that $(RG)^-$ is anticommutative. If $x \in G_*$ then $\sigma(x) \neq -1$.

Proof. If $x \in G_*$ and $\sigma(x) = -1$, then, for all $\alpha \in R$, the equation $\alpha(1 + \sigma(x)) = 0$ holds, so $\alpha x \in (RG)^-$. Thus, taking $\alpha = 1$, and $x \in (RG)^-$ we obtain that $\alpha x = x$ anticommutes with itself, so $x^2 + x^2 = 0$, which implies $2x^2 = 0$; a contradiction, for $\text{char}(R) \neq 2$. \square

The proofs of Lemmas 2.3 - 2.10 are quite similar to some Lemmas in [PT15], so we will just explain the details we can not point to this paper.

Lemma 2.3. If $(RG)^-$ is anticommutative, then $\text{char}(R) = 4$ or 8. Furthermore, if $*$ $\neq Id$, then $\text{char}(R) = 4$, $xx^* = x^*x$ and $x^2 \in G_*$, for all $x \notin G_*$.

Proof. The proof is analogous to [PT15, Lemmas 3.2 and 3.3]. \square

Corollary 2.4. If $(RG)^-$ is anticommutative, $*$ $\neq Id$, and $x \in G_*$, then there exists a nonzero $\alpha \in R$ such that $\alpha x \in (RG)^-$.

Proof. If $\sigma(x) = 1$, then we can easily verify $\alpha = 2$ satisfies such a claim, for $\text{char}(R) = 4$; if $\sigma(x) \neq 1$, then $\alpha = 1 - \sigma(x) \neq 0$ satisfies $\alpha(1 + \sigma(x)) = (1 - \sigma(x))(1 + \sigma(x)) = 1 - \sigma(x)^2$, and, for $x \in G_*$ and $x^*x^{-1} \in N$, we obtain $\sigma(x)^2 = 1$, so $\alpha(1 + \sigma(x)) = 0$. \square

Given $x, y \notin G_*$, as $(RG)^-$ is anticommutative, the following equation will play the same role as [PT15, equation (1)];

$$\begin{aligned} 0 &= (x - \sigma(x)x^*)(y - \sigma(y)y^*) + (y - \sigma(y)y^*)(x - \sigma(x)x^*) \\ &= xy + yx + \sigma(xy)x^*y^* + \sigma(xy)y^*x^* - \sigma(y)xy^* - \sigma(y)y^*x - \sigma(x)yx^* - \sigma(x)x^*y = 0. \end{aligned} \quad (1)$$

Lemma 2.5. Suppose that \mathcal{S} is anticommutative. Given $x, y \in G$, then, $xy = yx$ if, and only if, $x^*y = yx^*$. Moreover, if $x, y \notin G_*$, then $xy = yx = x^*y^* = y^*x^*$, $xy^* = y^*x = x^*y = yx^*$ and $2(1 + \sigma(xy)) = 2(\sigma(x) + \sigma(y)) = 0$.

Proof. The proof is analogous to [PT15, Lemma 3.4]. \square

Lemma 2.6. Suppose that $(RG)^-$ is anticommutative. If $x \notin G_*$ and $y \notin N \cup G_*$, then, $x^y \in \{x^*, x\}$.

Proof. We can prove this lemma analogously to [PT15, Lemma 3.5] replacing [PT15, equation (1)] with equation (1), except in case $xy = x^*y^*$ and $\sigma(xy) = -1$.

Suppose that $x^y \notin \{x, x^*\}$, $xy = x^*y^*$ and $\sigma(xy) = -1$. Notice that $(xy)^* = y^*x^* = yx$, then $xy + yx = xy - \sigma(xy)(xy)^* \in (RG)^-$, and

$$\begin{aligned} 0 &= (xy + yx)(x^{-1} - \sigma(x^{-1})x^{-*}) + (x^{-1} - \sigma(x^{-1})x^{-*})(xy + yx) \\ &= 2y + xyx^{-1} + x^{-1}yx - \sigma(x^{-1})(xyx^{-*} + yxx^{-*} + x^{-*}xy + x^{-*}yx), \end{aligned}$$

which implies $y \in \{xyx^{-1}, xyx^{-*}, yxx^{-*}, x^{-1}yx, x^{-*}xy, x^{-*}yx\}$. Since $x \notin G_*$ and $x^y \notin \{x, x^*\}$, it follows that $yx = x^*y$. As $yx = y^*x^*$, if $yx = x^*y$, we have $y^*x^* = x^*y$, and, applying involution, $xy = y^*x$; which leads to a contradiction by the proof of [PT15, Lemma 3.5]. \square

Lemma 2.7. *Suppose that $(RG)^-$ is anticommutative. If $x, y \notin G_*$, then it holds:*

- (i) $x^y \in \{x, x^*\}$. Particularly, $xy = x^*y^*$.
- (ii) $xy \in G_* \Leftrightarrow xy = yx$.
- (iii) If $xy \neq yx$, then $1 + \sigma(xy) = \sigma(x) + \sigma(y)$. If $xy = yx$, then $2(1 + \sigma(xy)) = 0 = 2(\sigma(x) + \sigma(y))$.

Proof. Suppose $(x, y) \neq 1$.

If $x, y \in N$, then, N satisfies (C) of Theorem 2.1, thus it has a unique nontrivial commutator s and $x^* = sx$, so $x^y = x(x, y) = xs = x^*$. Furthermore, $xy = xssy = x^*y^*$.

If $x, y \notin N$, then, applying Lemma 2.6 to x and y , we get $xy = yx^*$; moreover, applying the same lemma to y and x^* , we get $yx^* = x^*y^*$, in other words, $xy = x^*y^*$.

If $x \in N$ and $y \notin N$, applying Lemma 2.6 to x and y , as well as to x^* and y^* , we have $x^y = x^*$ and $x^*y^* = y^*x$. As $x^{-1} \in N \setminus G_*$ and $(x^{-1}, y) \neq 1$, then, $yx^{-1} = (x^{-1})^*y \neq (x^{-1})^*y^* = (yx^{-1})^*$, in other words, $yx^{-1} \notin G_*$; since $yx^{-1} \notin N$, applying the above case to y and yx^{-1} , it follows that $y^{x^{-1}} = y^{yx^{-1}} = y^*$, so $xy = y^*x = x^*y^*$.

If $x \notin N$ and $y \in N$, by the previous case, $yx = xy^* = y^*x^*$, and, applying involution, we find $x^*y^* = yx^* = xy$, so, $x^y = x^*$.

Finally, if $xy = yx$, then Lemma 2.5, guarantees (i).

The item (ii) follows as [PT15, Lemma 3.7 (ii)].

If $(x, y) \neq 1$, then applying item (i) to x and y , as well as to y and x , we have $xy = yx^* = y^*x = x^*y^*$; thus, item (iii) follows as [PT15, Lemma 3.6 (i) and (ii)], using equation (1) instead of [PT15, equation (1)]. \square

Lemma 2.8. *Suppose that $(RG)^-$ is anticommutative. Then, for all $y \in G_*$, $x \notin G_*$ and $\alpha \in R$ such that $\alpha y \in (RG)^-$, it follows that:*

- (i) $x^y \in \{x, x^*\}$.
- (ii) $xy \in G_* \Leftrightarrow xy \neq yx$.
- (iii) If $xy \neq yx$, then $\alpha\sigma(x) = \alpha$. If $xy = yx$, then $\alpha = -\alpha$.

Proof. The proof is analogous to [PT15, Lemma 3.9]. \square

Lemma 2.9. *Suppose that $(RG)^-$ is anticommutative. If $(G_*)' = \{1\}$, then G is abelian or a SLC-group with canonical involution $*$.*

Proof. Let $x \in G_*$ and $y \notin G_*$. If $xy \neq yx$, then, by item (ii) of Lemma 2.8, $xy \in G_*$; so, by hypothesis, $(x, xy) = 1$, thus $x(xy) = (xy)x$, which implies $xy = yx$, a contradiction; so $xy = yx$ for all $x \in G_*$ and $y \notin G_*$. As, according to the hypothesis, $(G_*)' = \{1\}$, then $G_* \subset \mathcal{Z}(G)$.

Let $(RG)_*$ be the spanning over R by the set $G_* \cup \{x + x^* : x \notin G_*\}$. We will prove that $(RG)_*$ is commutative. Since $G_* \subset \mathcal{Z}(G)$, it is enough to prove that $(x + x^*)$ commutes with $(y + y^*)$, $\forall x, y \notin G_*$.

If $xy = yx$, we can easily prove that $(x + x^*)(y + y^*) = (y + y^*)(x + x^*)$. Suppose that $xy \neq yx$, so,

$$\begin{aligned} (x + x^*)(y + y^*) &= xy + xy^* + x^*y + x^*y^* \\ &= yx^* + y^*x^* + yx + y^*x \quad (\text{by item (i) of Lemma 2.8}) \\ &= (y + y^*)(x + x^*). \end{aligned}$$

Thus $(RG)_*$ is commutative.

Let $*$: $RG \rightarrow RG$ be the linear extension of $*$: $G \rightarrow G$ to RG , and notice that $(RG)_*$ is the set of symmetric elements under $*$, so applying [JM06, Theorem 2.4], we conclude that G is abelian or a SLC-group with canonical involution $*$. \square

Lemma 2.10. *Suppose that $(RG)^-$ is anticommutative. Then, for all $x, y \in G_*$ and $\alpha, \beta \in R$ such that $\alpha x, \beta y \in (RG)^-$, it holds that:*

- (i) $xy = yx \Leftrightarrow xy \in G_*$.
- (ii) If $xy \neq yx$, it follows that $\alpha\beta = 0$; if $xy = yx$, then $2\alpha\beta = 0$.

Proof. This proof is analogous to [PT15, Lemma 3.10]. □

Proposition 2.11. *If $(RG)^-$ is anticommutative, then the following properties holds:*

- (i) If $*$ $\neq Id$, then exists $s \in \mathcal{Z}(G) \cap G_*$ such that $s^2 = 1$ and $x^* \in \{x, sx = xs\} \ \forall x \in G$.
- (ii) If G is nonabelian, then G has a unique nontrivial commutator $s \in \mathcal{Z}(G) \cap G_*$ (so it has order 2) and $x^* \in \{x, sx = xs\} \ \forall x \in G$.

Proof. (i) Suppose that $*$ $\neq Id$, thus $G \setminus G_* \neq \emptyset$. Let $x \notin G_*$ and $s = x^{-1}x^*$. Naturally $x^* = xs$ and since $(x, x^*) = 1$, then $x^* = sx$; furthermore, by Lemma 2.3, $s^2 = (x^{-1}x^*)^2 = x^{-2}(x^*)^2 = x^{-2}x^2 = 1$ and $s = xx^{-*}$. Finally, if $y \notin G_*$, then, by Lemma 2.7 (i), $xy = x^*y^*$, in other words, $s = xx^{-*} = y^{-1}y^*$; thus s does not depend on x , so $y^* \in \{y, sy = ys\} \ \forall y \in G$.

Suppose that $(s, x) \neq 1$, for some $x \in G$, then $sx^{-1}sx = (s, x) = s$; thus $s^x = 1$, a contradiction; so, $s \in \mathcal{Z}(G)$. Furthermore, if $s \notin G_*$, then $s^* = s^2 = 1$, also a contradiction, since $1^* = 1 \neq s$.

(ii) If G is nonabelian, then the identity does not define an involution on G , so $Id \neq *$, thus, by (i), if $x \notin G_*$, then $x^* = xs = sx$.

Notice that, if $x \notin G_*$ and $y \in G$ satisfy $(x, y) \neq 1$, then, by Lemma 2.7 (i) or Lemma 2.8 (i), we obtain that $x^y = x^*$, so, $(x, y) = x^{-1}x^y = x^{-1}x^* = x^{-1}xs = s$.

If $x, y \in G_*$ and $(x, y) \neq 1$, by Lemma 2.10 (i), $xy \notin G_*$; hence, by item (i), $yx = y^*x^* = (xy)^* = sxy$, in other words $(x, y) = s$. □

Theorem 2.12. *Let R be a commutative ring with $\text{char}(R) \neq 2$, let G be group with involution $*$ and let $\sigma : G \rightarrow \mathcal{U}(R)$ be a nontrivial orientation which is compatible with $*$ in the sense that $\sigma(xx^*) = 1$, for all $x \in G$ and $N = \ker \sigma$. Then, $(RG)^-$ is anticommutative if, and only if, the following holds:*

- (i) One of the these three possibilities holds:
 - (a) $\text{char}(R) = 4$ or 8 , G is abelian, and $*$ $= Id_G$.
 - (b) $\text{char}(R) = 4$, G is abelian, $*$ $\neq Id_G$, and exists $s \in G_*$, such that $s^2 = 1$ and $x^* \in \{x, xs\}$, $\forall x \in G$.
 - (c) $\text{char}(R) = 4$, G has an unique nontrivial commutator s (thus $s \in G_* \cap \mathcal{Z}(G)$) and $x^* \in \{x, xs\}$.
- (ii) $\forall x, y \notin G_*$, if $xy \neq yx$, then $1 + \sigma(xy) = \sigma(x) + \sigma(y)$; if $xy = yx$, then $2(1 + \sigma(xy)) = 0 = 2(\sigma(x) + \sigma(y))$.
- (iii) $\forall x \notin G_*$, $y \in G_*$, and $\alpha \in R$ with $\alpha y \in (RG)^-$. If $xy \neq yx$, then $\alpha\sigma(x) = \alpha$; if $xy = yx$, then $\alpha = -\alpha$.
- (iv) $\forall x, y \in G_*$ and $\alpha, \beta \in R$ with $\alpha x, \beta y \in (RG)^-$. If $xy \neq yx$, then $\alpha\beta = 0$; if $xy = yx$, then $2\alpha\beta = 0$.

Furthermore, if $(G_*)' = \{1\}$ and G is nonabelian, then G is a SLC-group with canonical involution $*$.

Proof. Suppose that $(RG)^-$ is anticommutative. By Lemma 2.3, we conclude that $\text{char}(R) = 4$ or 8 . If $* = \text{Id}_G$, then G is abelian and (a) holds. Suppose $* \neq \text{Id}$, then, by Lemma 2.3, $\text{char}(R) = 4$.

Using Proposition 2.11 we can easily verify that, if G is abelian, item (i) implies (b), and case contrary, (c) follows from item (ii).

To verify items (ii)-(iv), it is enough apply item (iii), (iii) e (ii), of Lemma 2.7, 2.8 and 2.10, respectively.

The converse could be proved in the same way of converse of [PT15, Theorem 3.15].

Finally, if $(G_*)' = \{1\}$ and G is nonabelian, then Lemma 2.9 guaranties that G is a SLC-group with canonical involution $*$. \square

Corollary 2.13 (Theorem 2.1 [GP13]). *Let R be a commutative ring with $\text{char}(R) \neq 2$, let G be group with involution $*$ and let $\sigma : G \rightarrow \{\pm 1\}$ be a nontrivial orientation which is compatible with $*$ in the sense that $\sigma(xx^*) = 1$, for all $x \in G$ and $N = \ker \sigma$. Then, $(RG)^-$ is anticommutative if, and only if, $\text{char}(R) = 4$ and the following holds:*

- (1) G is abelian, $*_N = \text{Id}_N$ and exists $s \in N_*$ such that $x^* = xs$, $\forall x \notin N$.
- (2) G is a SLC-group with canonical involution $*$ and $x^* = xs$, $\forall x \notin N$.

Proof. Suppose $(RG)^-$ is anticommutative. By Lemma 2.2, $G_* \setminus N = \emptyset$, so $* \neq \text{Id}$. Applying Theorem 2.12, we obtain that G is abelian or has an unique nontrivial commutator s and $x^* = sx$, $\forall x \notin G_*$. In particular, as $G_* \setminus N = \emptyset$, $s \in N$ and $x^* = sx$, $\forall x \notin N$.

By Theorem 2.1 we have three possibilities to N , namely (A), (B) or (C).

Suppose G is abelian, so (C) does not occur. Let us prove that (B) also does not hold, leading to (A), hence item (1) happens.

If (B) holds, then exists $x \in N$ such that $x^* = sx$. Taking $y \notin N$, we obtain that $y^* = sy$, moreover, $xy \notin N$, which implies $xy \notin G_*$, since $G_* \setminus N = \emptyset$. As $x, y, xy \notin G_*$, by Lemma 2.7 (ii), $xy \neq yx$, so G is nonabelian, a contradiction. Thus (1) holds.

Suppose that G is nonabelian. By Theorem 2.12 and the fact that $G_* \subset N$, in order to find (2), it is enough prove that $\{1\} = (G_*)' = (N_*)'$. Notice that this condition holds if N is abelian, in other words, if (A) or (B) occur. Suppose that (C) holds and let $x, y \in N_*$; if $xy \neq yx$, by item (ii) of Lemma 2.10, $xy \notin G_* \subset N$, a contradiction, for $x, y \in N$. So, (2) holds.

Conversely, notice that items (i) and (ii) of Theorem 2.12 can be easily verified. Notice that, if $x \in G_*$ and $\alpha x \in (RG)^-$ with $\alpha \neq 0$, then $\alpha(1 + \sigma(x)) = 0$, in other words $2\alpha = 0$; so, if G is abelian or a SLC-group, then $G_* \subset \mathcal{Z}(G)$, thus, (iii) and (iv) of Theorem 2.12 hold, so $(RG)^-$ is anticommutative. \square

Proposition 2.14. *Let R be a commutative ring with $\text{char}(R) \neq 2$, let G be group with involution $*$ and let $\sigma : G \rightarrow \mathcal{U}(R)$ be a nontrivial orientation which is compatible with $*$ in the sense that $\sigma(xx^*) = 1$, for all $x \in G$, with $\exp(G/N) = 2$ and $N = \ker \sigma$ satisfying (1) of Theorem 2.1. If $G_* \subset N$, then $(RG)^-$ is anticommutative if, and only if, $\text{char}(R) = 4$ and one of the following holds:*

- (i) G is abelian and $\sigma = \{\pm 1\}$.
- (ii) (a) G is a SLC-group with canonical involution $*$ and $G/N \simeq C_2 \times C_2$.
 - (b) $\forall x, y \notin G_*$, if $xy \neq yx$, then $1 + \sigma(xy) = \sigma(x) + \sigma(y)$; if $xy = yx$, then $2(1 + \sigma(xy)) = 0 = 2(\sigma(x) + \sigma(y))$.
 - (c) $2\alpha = 0$, $\forall \alpha \in R$ with $\alpha x \in (RG)^-$.

Proof. Suppose that $(RG)^-$ is anticommutative. By Theorem 2.12, we have $\text{char}(R) = 4$. Furthermore, since N satisfies (1), N is abelian and $N \subset G_*$, so, if $G_* \subset N$, then, $G_* = N$; as N is abelian, $(G_*)' = \{1\}$, then, by Theorem 2.12, we obtain that G is abelian or a SLC-group with canonical involution $*$, thus $N = G_* \subset \mathcal{Z}(G)$.

Suppose that G is abelian and let $x, y \notin G_* = N$. By item (ii) of Lemma 2.7 we obtain that $xy = yx \Leftrightarrow xy \in N \Leftrightarrow xN = y^{-1}N = yN$, where the last identity follows due to $\exp(G/N) = 2$. This way, as G is abelian, any element that is not in N , is in the class xN , so, N has only two cosets in G , which implies that $G/N \simeq C_2$, in other words $\sigma = \{\pm 1\}$. Thus (i) holds.

Suppose now that G is nonabelian, that is, G is a SLC-group. If G is a SLC-group, then $N = G_* = \mathcal{Z}(G)$, so, by [JM06, Theorem 2.4], we have that $G/N = G/\mathcal{Z}(G) \simeq C_2 \times C_2$, and (a) holds.

To verify (b), is enough to apply item (ii) of Theorem 2.12. Notice that, if $\alpha x \in (RG)^-$, then $x \in G_* = \mathcal{Z}(G)$, so, applying item (iii) of Theorem 2.12, we conclude (c).

Conversely, suppose that (i) holds. In this case, σ is a classic orienatation, then, by Corolary 2.13, we have that $(RG)^-$ is anticommutative.

Finally, suppose that (ii) holds. We will prove that item (i)-(iv) of Theorem 2.12 occur. Item (i) naturally is verified by SLC-groups, and item (ii) is exactly item (b). Observe that, if $G_* = \mathcal{Z}(G)$ and $\alpha x \in (RG)^-$, we have that $xy = yx, \forall y \in G$. Thus, hyphotesis (c) guaranties (iii) and (iv). \square

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